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J. Math. Anal. Appl. 326 (2007) 556–569

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Oscillation of the solutions of parabolic equations with nonlinear neutral terms

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Received 22 April 2005

Available online 17 April 2006

Submitted by J. Lavery

Abstract

In the present paper the oscillatory properties of the solutions of parabolic equations with nonlinear neutral terms are investigated. Our approach is to reduce the multi-dimensional problem to a one-dimensional problem for delay differential inequalities.

Keywords: Oscillation; Parabolic equation; Nonlinear neutral terms

1. Introduction

The scalar autonomous ordinary differential equation

$$\frac{dN(t)}{dt} = rN(t)\left[1 - \frac{N(t)}{K}\right],$$

where $r, K \in (0, \infty)$, is known as the logistic equation in mathematical ecology. This equation can be rewritten in a more general following form:

$$\frac{dN(t)}{dt} = rN(t)\left[\left(1 - \frac{N(t-\sigma)}{K}\right) + c\frac{d}{dt}\left(1 - \frac{N(t-\sigma)}{K}\right)\right] \tag{$*$}$$

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.09.048

in which $c \in \mathbb{R}$ and $\sigma \in (0, \infty)$. By the change of variable $x(t) = \ln(N(t)/K)$, Eq. $(*)$ is made equivalent to

$$\frac{d}{dt}\left[x(t) + rc\left(e^{x(t-\sigma)} - 1\right)\right] + r\left(e^{x(t-\sigma)} - 1\right) = 0,$$

which was studied in Gopalsamy and Zhang [1]. Sufficient conditions for oscillation of solutions of neutral delay logistic differential equations were obtained by Györi and Ladas [2] in the case $c = -1$.

In [3], the author investigated the stability of zero solution of the more general nonlinear neutral delay differential equation

$$\frac{d}{dt}\left(x(t) + h(t)\omega\left(x(t-\rho)\right)\right) + q(t)\varphi\left(x(t-\sigma)\right) = 0$$

and obtained the first 3/2 stability results of neutral delay differential equations in the literature.

The oscillation of the parabolic differential equations of neutral type has been studied by some authors; for example, see [4,5], etc. The purpose of this paper is to establish sufficient conditions for oscillations of the following equation by using the results of Yoshida [7] and Shoukaku and Yoshida [6].

We are concerned with the oscillation of solutions of the parabolic equation

$$\frac{\partial}{\partial t}\left(u(x,t) + \sum_{i=1}^{l} h_i(t)\omega_i\left(u\left(x, \rho_i(t)\right)\right)\right) - a(t)\Delta u(x,t) - \sum_{i=1}^{k} b_i(t)\Delta u\left(x, \tau_i(t)\right)$$
$$+ c\left(x, t, \left(z_i[u](x,t)\right)_{i=1}^{\tilde{m}}\right) = 0, \quad (x,t) \in G \times (0,\infty) \equiv \Omega, \tag{E}$$

where Δ is the Laplacian in $\mathbb{R}^n$ and G is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary ∂G.

We assume throughout this paper that:

(H1) $h_i(t) \in C^1([0,\infty); [0,\infty))$ $(i = 1, 2, \ldots, l)$,
$a(t) \in C([0,\infty); [0,\infty))$,
$b_i(t) \in C([0,\infty); [0,\infty))$ $(i = 1, 2, \ldots, k)$;

(H2) $\rho_i(t) \in C([0,\infty); [0,\infty))$, $\lim_{t\to\infty} \rho_i(t) = \infty$ $(i = 1, 2, \ldots, l)$,
$\tau_i(t) \in C([0,\infty); [0,\infty))$, $\lim_{t\to\infty} \tau_i(t) = \infty$ $(i = 1, 2, \ldots, k)$;

(H3) $\omega_i(s) \in C(\mathbb{R}; \mathbb{R})$ $(i = 1, 2, \ldots, l)$,
$\omega_i(-s) = -\omega_i(s)$, $\omega_i(s)$ are positive and concave in $(0,\infty)$;

(H4) there are positive constants α_i such that

$$\omega_i(s) \leqslant \alpha_i s \quad (i = 1, 2, \ldots, l) \text{ for all } s > 0;$$

(H5) $\sum_{i=1}^{l} \alpha_i h_i(t) \leqslant 1$;

(H6) $t \leqslant \rho_i(t)$ $(i = 1, 2, \ldots, l)$;

(H7) $c(x, t, (\xi_i)_{i=1}^{\tilde{m}}) \in (\overline{\Omega} \times \mathbb{R}^{\tilde{m}}; \mathbb{R})$,
$c(x, t, (\xi_i)_{i=1}^{\tilde{m}}) \geqslant \sum_{i=1}^{m} q_i(t)\varphi_i(\xi_i)$ for $(x, t, (\xi_i)_{i=1}^{\tilde{m}}) \in \Omega \times [0,\infty)^{\tilde{m}}$,
$c(x, t, (-\xi_i)_{i=1}^{\tilde{m}}) \leqslant -\sum_{i=1}^{m} q_i(t)\varphi_i(\xi_i)$ for $(x, t, (\xi_i)_{i=1}^{\tilde{m}}) \in \Omega \times [0,\infty)^{\tilde{m}}$,
where $[0,\infty)^j = [0,\infty) \times [0,\infty)^{j-1}$ $(j = 1, 2, \ldots, \tilde{m})$, $q_i(t) \in C([0,\infty); [0,\infty))$, $\varphi_i(\xi) \in C([0,\infty); [0,\infty))$, and $\varphi_i(\xi)$ are convex in $(0,\infty)$ $(i = 1, 2, \ldots, m)$;

(H8) $$z_i[u](x,t) = \begin{cases} u(x, \sigma_i(t)) & (i = 1, 2, \dots, m), \\ \max_{s \in B_i(t)} u(x, s) & (i = m+1, m+2, \dots, m_1), \\ \sum_{j=1}^{N_i} \int_G K_{ij}(x, t, y)\phi_{ij}(u(y, \sigma_{ij}(t)))\, dy & \\ & (i = m_1+1, m_1+2, \dots, \tilde{m}), \end{cases}$$

where $\sigma_i(t) \in C([0,\infty); \mathbb{R})$ $(i = 1, 2, \dots, m)$, $\lim_{t\to\infty} \sigma_i(t) = \infty$, $B_i(t)$ $(i = m+1, m+2, \dots, m_1)$ are closed bounded sets of $[0,\infty)$ such that $\lim_{t\to\infty} \max_{s\in B_i(t)} s = \infty$, $K_{ij}(x,t,y) \in C(\overline{\Omega} \times \overline{G}; [0,\infty))$, $\sigma_{ij}(t) \in C([0,\infty); \mathbb{R})$ $(i = m_1+1, m_1+2, \dots, \tilde{m};\ j = 1, 2, \dots, N_i)$, $\lim_{t\to\infty} \sigma_{ij}(t) = \infty$, and $\phi_{ij}(z) \in C(\mathbb{R}; \mathbb{R})$ are odd functions with the property that $\phi_{ij}(z) \geqslant 0$ for $z > 0$.

We consider two kinds of boundary conditions:

(B_1) $u = 0$ on $\partial G \times [0,\infty)$,
(B_2) $\partial u/\partial \nu + \mu u = 0$ on $G \times [0,\infty)$,

where ν is the unit exterior normal vector to ∂G and μ is a nonnegative continuous function on $\partial G \times [0,\infty)$.

Definition 1. By a *solution* of Eq. (E) we mean a function $u(x,t) \in C^2(\overline{G} \times [t_{-1},\infty); \mathbb{R}) \cap C^1(\overline{G} \times [\hat{t}_{-1},\infty); \mathbb{R}) \cap C(\overline{G} \times [\tilde{t}_{-1},\infty); \mathbb{R})$ which satisfies (E), where

$$t_{-1} = \min\Big\{0, \min_{1\leqslant i\leqslant k}\Big\{\inf_{t\geqslant 0} \tau_i(t)\Big\}\Big\},$$
$$\hat{t}_{-1} = \min\Big\{0, \min_{1\leqslant i\leqslant l}\Big\{\inf_{t\geqslant 0} \rho_i(t)\Big\}\Big\},$$
$$\tilde{t}_{-1} = \min\Big\{0, \min_{1\leqslant i\leqslant m}\Big\{\inf_{t\geqslant 0} \sigma_i(t)\Big\}, \min_{\substack{m_1+1\leqslant i\leqslant \tilde{m} \\ 1\leqslant j\leqslant N_i}}\Big\{\inf_{t\geqslant 0} \sigma_{ij}(t)\Big\}\Big\}.$$

Definition 2. A solution u of Eq. (E) is said to be *oscillatory* in Ω if u has zero in $G \times (t,\infty)$ for any $t > 0$.

2. Oscillation results for equation (E)

The object of this section is to reduce oscillation problems for (E) to oscillation problems for functional differential inequalities.

The first eigenvalue λ_0 of the eigenvalue problem

$$\Delta w + \lambda w = 0 \quad \text{in } G,$$
$$w = 0 \quad \text{on } \partial G$$

is positive and the corresponding eigenfunction $\Phi(x)$ may be chosen so that $\Phi(x) > 0$ in G.

Associated with a function $u \in C^2(\Omega; \mathbb{R}) \cap C^1(\overline{\Omega}; \mathbb{R})$, we define

$$U(t) = \frac{1}{\int_G \Phi(x)\, dx} \int_G u(x,t)\Phi(x)\, dx,$$
$$\widetilde{U}(t) = \frac{1}{|G|} \int_G u(x,t)\, dx,$$

where $|G| = \int_G dx$.

Theorem 1. *Assume that* (H1)–(H8) *hold, and that*

(H9) *there exists an integer $j \in \{1, 2, \dots, m\}$ such that*

$$\varphi_j(s_1 s_2) \geqslant \varphi_{j1}(s_1)\varphi_{j2}(s_2)$$

for $s_1 > 0$, $s_2 > 0$, where $\varphi_{j1}(s_1) \geqslant 0$, $\varphi_{j2}(s_2) > 0$ and $\varphi_{j2}(s_2)$ is nondecreasing for $s_2 > 0$.

If every eventually positive solution $y(t)$ of the differential inequality

$$y'(t) + q_j(t)\varphi_{j1}\left(1 - \sum_{i=1}^{l} \alpha_i h_i\bigl(\sigma_j(t)\bigr)\right)\varphi_{j2}\bigl(y\bigl(\sigma_j(t)\bigr)\bigr) \leqslant 0 \tag{1}$$

satisfies $\lim_{t\to\infty} y(t) = 0$, then every solution u of the problem (E), (B_1) *is oscillatory in Ω or satisfies*

$$\lim_{t\to\infty} U(t) = 0. \tag{2}$$

Proof. Suppose that there exists a nonoscillatory solution u which does not satisfy (2). We assume that $u > 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$. Since (H2) holds, we see that $u(x, \rho_i(t)) > 0$ ($i = 1, 2, \dots, l$), $u(x, \tau_i(t)) > 0$ ($i = 1, 2, \dots, k$), $u(x, \sigma_i(t)) > 0$ ($i = 1, 2, \dots, m$) and $u(x, \sigma_{ij}(t)) > 0$ ($i = m_1 + 1, m_1 + 2, \dots, \tilde{m}$) in $G \times [t_1, \infty)$ for some $t_1 \geqslant t_0$. The hypothesis (H7) implies that

$$c\bigl(x, t, \bigl(z_i[u](x, t)\bigr)_{i=1}^{\tilde{m}}\bigr) \geqslant \sum_{i=1}^{m} q_i(t)\varphi_i\bigl(u\bigl(x, \sigma_i(t)\bigr)\bigr) \quad \text{in } G \times [t_1, \infty).$$

Hence, from (E) we can see that

$$\begin{aligned} &\frac{\partial}{\partial t}\left(u(x, t) + \sum_{i=1}^{l} h_i(t)\omega_i\bigl(u\bigl(x, \rho_i(t)\bigr)\bigr)\right) - a(t)\Delta u(x, t) - \sum_{i=1}^{k} b_i(t)\Delta u\bigl(x, \tau_i(t)\bigr) \\ &\quad + \sum_{i=1}^{m} q_i(t)\varphi_i\bigl(u\bigl(x, \sigma_i(t)\bigr)\bigr) \leqslant 0 \quad \text{in } G \times [t_1, \infty). \end{aligned} \tag{3}$$

We set

$$z(x, t) = u(x, t) + \sum_{i=1}^{l} h_i(t)\omega_i\bigl(u\bigl(x, \rho_i(t)\bigr)\bigr) \quad \text{in } G \times [t_1, \infty). \tag{4}$$

From (E) and (4), we obtain

$$\begin{aligned} &\frac{\partial}{\partial t}\bigl(z(x, t)\bigr) - a(t)\Delta u(x, t) - \sum_{i=1}^{k} b_i(t)\Delta u\bigl(x, \tau_i(t)\bigr) + \sum_{i=1}^{m} q_i(t)\varphi_i\bigl(u\bigl(x, \sigma_i(t)\bigr)\bigr) \leqslant 0 \\ &\quad \text{in } G \times [t_1, \infty). \end{aligned} \tag{5}$$

Multiplying (5) by $\Phi(x)(\int_G \Phi(x)\,dx)^{-1}$ and integrating over G, we obtain

$$\begin{aligned} &z'(t) - a(t)K_\Phi \int_G \Delta u(x, t)\Phi(x)\,dx - \sum_{i=1}^{k} b_i(t)K_\Phi \int_G \Delta u\bigl(x, \tau_i(t)\bigr)\Phi(x)\,dx \\ &\quad + \sum_{i=1}^{m} q_i(t)K_\Phi \int_G \varphi_i\bigl(u\bigl(x, \sigma_i(t)\bigr)\bigr)\Phi(x)\,dx \leqslant 0, \quad t \geqslant t_1, \end{aligned} \tag{6}$$

where

$$K_\Phi = \left(\int_G \Phi(x)\,dx\right)^{-1}, \qquad z(t) = \frac{1}{\int_G \Phi(x)\,dx} \int_G z(x,t)\Phi(x)\,dx.$$

It follows from Green's formula that

$$K_\Phi \int_G \Delta u(x,t)\Phi(x)\,dx = -\lambda_1 K_\Phi \int_G u(x,t)\Phi(x)\,dx = -\lambda_1 U(t), \quad t \geqslant t_1. \tag{7}$$

Analogously we obtain

$$K_\Phi \int_G \Delta u\bigl(x, \tau_i(t)\bigr)\Phi(x)\,dx = -\lambda_1 U\bigl(\tau_i(t)\bigr), \quad t \geqslant t_1. \tag{8}$$

An application of Jensen's inequality shows that

$$K_\Phi \int_G \varphi_i\bigl(u\bigl(x, \sigma_i(t)\bigr)\bigr)\Phi(x)\,dx \geqslant \varphi_i\bigl(U\bigl(\sigma_i(t)\bigr)\bigr), \quad t \geqslant t_1. \tag{9}$$

Combining (6)–(9) yields

$$z'(t) + \lambda_1 a(t)U(t) + \sum_{i=1}^{l} b_i(t)U\bigl(\tau_i(t)\bigr) + \sum_{i=1}^{m} q_i(t)\varphi_i\bigl(U\bigl(\sigma_i(t)\bigr)\bigr) \leqslant 0, \quad t \geqslant t_1.$$

Since $U(t)$ is eventually positive, the above inequality implies

$$z'(t) + q_j(t)\varphi_j\bigl(U\bigl(\sigma_j(t)\bigr)\bigr) \leqslant 0, \quad t \geqslant t_1. \tag{10}$$

Since

$$z'(t) \leqslant -q_j(t)\varphi_j\bigl(U\bigl(\sigma_j(t)\bigr)\bigr) \leqslant 0, \quad t \geqslant t_1,$$

thus $z(t)$ is nonincreasing.

On the other hand, multiplying (4) by $\Phi(x)(\int_G \Phi(x)\,dx)^{-1}$, integrating over G and using the hypothesis (H3), we obtain

$$z(t) \leqslant U(t) + \sum_{i=1}^{l} h_i(t)\omega_i\bigl(U\bigl(\rho_i(t)\bigr)\bigr).$$

Then

$$z(t) - \sum_{i=1}^{l} h_i(t)\omega_i\bigl(U\bigl(\rho_i(t)\bigr)\bigr) \leqslant U(t), \quad t \geqslant t_1.$$

Since (H4) holds, we see that

$$U(t) \geqslant z(t) - \sum_{i=1}^{l} h_i(t)\omega_i\bigl(U\bigl(\rho_i(t)\bigr)\bigr) \geqslant z(t) - \sum_{i=1}^{l} h_i(t)\alpha_i U\bigl(\rho_i(t)\bigr), \quad t \geqslant t_2 \tag{11}$$

for some $t_2 \geqslant t_1$. From (4) we have

$$z(x,t) \geqslant u(x,t), \quad t \geqslant t_2. \tag{12}$$

Multiplying (12) by $\Phi(x)(\int_G \Phi(x)\,dx)^{-1}$ and integrating over G, we obtain

$$z(t) \geqslant U(t) > 0, \quad t \geqslant t_2. \tag{13}$$

Using (11) and (13) yields

$$U(t) \geqslant z(t) - \sum_{i=1}^{l} \alpha_i h_i(t) z\big(\rho_i(t)\big), \quad t \geqslant t_2. \tag{14}$$

Since $z(t)$ is nonincreasing, from (H6) we can find that

$$z\big(\rho_i(t)\big) \leqslant z(t), \quad t \geqslant t_2. \tag{15}$$

Combining (14) with (15) yields

$$U(t) \geqslant \left(1 - \sum_{i=1}^{l} \alpha_i h_i(t)\right) z(t), \quad t \geqslant t_2. \tag{16}$$

Applying (10) and (16), we obtain

$$z'(t) + q_j(t)\varphi_j\left(\left(1 - \sum_{i=1}^{l} \alpha_i h_i\big(\sigma_j(t)\big)\right) z\big(\sigma_j(t)\big)\right) \leqslant 0, \quad t \geqslant t_2,$$

which can be rewritten from (H9) as

$$z'(t) + q_j(t)\varphi_{j1}\left(1 - \sum_{i=1}^{l} \alpha_i h_i\big(\sigma_j(t)\big)\right) \varphi_{j2}\big(z\big(\sigma_j(t)\big)\big) \leqslant 0, \quad t \geqslant t_2.$$

In view of inequality (13) and the first assumption, we find that $z(t)$ is a positive solution of (1) which does not satisfy $\lim_{t\to\infty} z(t) = 0$. This contradicts the hypothesis.

If $u < 0$ in $G \times [t_0, \infty)$, it can be shown that

$$c\big(x, t, \big(z_i[u]\big)_{i=1}^{\tilde{m}}\big) \leqslant -\sum_{i=1}^{m} q_i(t)\varphi_i\big(-u\big(x, \sigma_i(t)\big)\big) \quad \text{in } G \times [t_1, \infty)$$

for some $t_1 \geqslant t_0$. Letting $v = -u$, we obtain

$$\frac{\partial}{\partial t}\left(v(x,t) + \sum_{i=1}^{l} h_i(t)\omega_i\big(v\big(x, \rho_i(t)\big)\big)\right) - a(t)\Delta v(x,t) - \sum_{i=1}^{k} b_i(t)\Delta v\big(x, \tau_i(t)\big)$$
$$+ \sum_{i=1}^{m} q_i(t)\varphi_i\big(v\big(x, \sigma_i(t)\big)\big) \leqslant 0, \quad \text{in } G \times [t_1, \infty).$$

Proceeding as in the case where $u > 0$, we are led to a contradiction. This completes the proof. □

Theorem 2. *Assume that* (H1)–(H9) *hold. If every eventually positive solution $y(t)$ of the differential inequality* (1) *satisfies* $\lim_{t\to\infty} y(t) = 0$*, then every solution u of the problem* (E), (B_2) *is oscillatory in Ω or satisfies*

$$\lim_{t\to\infty} \widetilde{U}(t) = 0. \tag{17}$$

Proof. Assume on the contrary, that there exists a solution u of the problem (E), (B_2) such that $u > 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$ which does not satisfy (17). By the hypothesis (H2) we have $u(x, \rho_i(t)) > 0$ ($i = 1, 2, \ldots, l$), $u(x, \tau_i(t)) > 0$ ($i = 1, 2, \ldots, k$), $u(x, \sigma_i(t)) > 0$ ($i = 1, 2, \ldots, m$) and $u(x, \sigma_{ij}(t)) > 0$ ($i = m_1 + 1, m_1 + 2, \ldots, \tilde{m}$) in $G \times [t_1, \infty)$ for some $t_1 \geqslant t_0$. Arguing as in the proof of Theorem 1, we see that inequality (3) holds for $t \geqslant t_1$. Let $z(x, t)$ be defined as in (4). We obtain inequality (5). Dividing (5) by $|G|$ and then integrating over G yields

$$\tilde{z}'(t) - \frac{a(t)}{|G|} \int_G \Delta u(x, t)\, dx - \sum_{i=1}^{k} \frac{b_i(t)}{|G|} \int_G \Delta u\bigl(x, \tau_i(t)\bigr)\, dx + \sum_{i=1}^{m} \frac{q_i(t)}{|G|} \int_G \varphi_i\bigl(u\bigl(x, \sigma_i(t)\bigr)\bigr)\, dx \leqslant 0, \quad t \geqslant t_1, \tag{18}$$

where

$$\tilde{z}(t) = \frac{1}{|G|} \int_G z(x, t)\, dx.$$

From Green's formula it follows that

$$\frac{1}{|G|} \int_G \Delta u(x, t)\, dx = -\frac{1}{|G|} \int_{\partial G} \bigl(\mu(x, t) u(x, t)\bigr)\, dS \leqslant 0, \quad t \geqslant t_1. \tag{19}$$

Analogously we obtain

$$\frac{1}{|G|} \int_G \Delta u\bigl(x, \tau_i(t)\bigr)\, dx = -\frac{1}{|G|} \int_{\partial G} \bigl(\mu\bigl(x, \tau_i(t)\bigr) u\bigl(x, \tau_i(t)\bigr)\bigr)\, dS \leqslant 0, \quad t \geqslant t_1. \tag{20}$$

An application of Jensen's inequality shows that

$$\frac{1}{|G|} \int_G \varphi_i\bigl(u\bigl(x, \sigma_i(t)\bigr)\bigr)\, dx \geqslant \varphi_i\bigl(\widetilde{U}\bigl(\sigma_i(t)\bigr)\bigr), \quad t \geqslant t_1. \tag{21}$$

Combining (18)–(21) yields

$$\tilde{z}'(t) + \sum_{i=1}^{m} q_i(t) \varphi_i\bigl(\widetilde{U}\bigl(\sigma_i(t)\bigr)\bigr) \leqslant 0, \quad t \geqslant t_1,$$

or

$$\tilde{z}'(t) + q_j(t) \varphi_j\bigl(\widetilde{U}\bigl(\sigma_j(t)\bigr)\bigr) \leqslant 0, \quad t \geqslant t_1. \tag{22}$$

Since

$$\tilde{z}'(t) \leqslant -q_j(t) \varphi_j\bigl(\widetilde{U}\bigl(\sigma_j(t)\bigr)\bigr) \leqslant 0, \quad t \geqslant t_1,$$

therefore $\tilde{z}(t)$ is nonincreasing. Dividing (4) by $|G|$ and integrating over G, from (H3) we have

$$\tilde{z}(t) \leqslant \widetilde{U}(t) + \sum_{i=1}^{l} h_i(t) \omega_i\bigl(\widetilde{U}\bigl(\tau_i(t)\bigr)\bigr). \tag{23}$$

Next, dividing (12) by $|G|$ and then integrating over G, we obtain

$$\tilde{z}(t) \geqslant \widetilde{U}(t) > 0, \quad t \geqslant t_1. \tag{24}$$

Combining (23) with (24) yields

$$\widetilde{U}(t) \geqslant \tilde{z}(t) - \sum_{i=1}^{l} h_i(t)\omega_i\big(\tilde{z}\big(\rho_i(t)\big)\big) = \left(1 - \sum_{i=1}^{l} h_i(t)\frac{\omega_i(\tilde{z}(\rho_i(t)))}{\tilde{z}(t)}\right)\tilde{z}(t), \quad t \geqslant t_1.$$

Using (H4) and $\tilde{z}'(t) \leqslant 0$, the above inequality implies

$$\widetilde{U}(t) \geqslant \left(1 - \sum_{i=1}^{l} h_i(t)\frac{\omega_i(\tilde{z}(\rho_i(t)))}{\tilde{z}(\rho_i(t))}\right)\tilde{z}(t) \geqslant \left(1 - \sum_{i=1}^{l} \alpha_i h_i(t)\right)\tilde{z}(t), \quad t \geqslant t_2 \tag{25}$$

for some $t_2 \geqslant t_1$. Substituting (25) into (22), we have

$$\tilde{z}'(t) + q_j(t)\varphi_j\left(\left(1 - \sum_{i=1}^{l} \alpha_i h_i\big(\sigma_j(t)\big)\right)\tilde{z}\big(\sigma_j(t)\big)\right) \leqslant 0, \quad t \geqslant t_1.$$

From (H9) we can see that

$$\tilde{z}'(t) + q_j(t)\varphi_{j1}\left(1 - \sum_{i=1}^{l} \alpha_i h_i\big(\sigma_j(t)\big)\right)\varphi_{j2}\big(\tilde{z}\big(\sigma_j(t)\big)\big) \leqslant 0, \quad t \geqslant t_1.$$

Hence, $\tilde{z}(t)$ is a positive solution of (1) which does not satisfy $\lim_{t\to\infty} \tilde{z}(t) = 0$. This contradicts the hypothesis. The case where $u < 0$ can be treated similarly, and we are led to a contradiction. The proof is complete. □

Remark 1. If $U(t)$ is eventually positive, then the inequality

$$0 \leqslant U(t) \leqslant z(t)$$

holds. Therefore, $\lim_{t\to\infty} z(t) = 0$ implies (2). In a same way we can see that (17) of Theorem 2 holds.

Analogously to the proof of [6, Corollaries 1 and 2], or [7, Corollaries 1 and 2], we obtain the following.

Corollary 1. *Assume that* (H1)–(H9) *hold. If*

$$\int_{R[\sigma_j]} q_j(t)\varphi_{j1}\left(1 - \sum_{i=1}^{l} \alpha_i h_i\big(\sigma_j(t)\big)\right) dt = \infty, \tag{26}$$

then every solution u of the problem (E), (B$_1$) *is oscillatory in Ω or satisfies* (2)*, where* $R[\sigma_j] = \{t \in [0,\infty):\ 0 \leqslant \sigma_j(t) \leqslant t\}$.

Corollary 2. *Assume that* (H1)–(H9) *hold. If* (26) *holds, then every solution u of the problem* (E)*,* (B$_2$) *is oscillatory in Ω or satisfies* (17)*.*

3. Oscillation results for the linear case of equation (E)

In the linear case of (E), we consider the parabolic equation of the form

$$\frac{\partial}{\partial t}\left(u(x,t)+\sum_{i=1}^{l}h_i(t)\omega_i\big(u\big(x,\rho_i(t)\big)\big)\right)-a(t)\Delta u(x,t)-\sum_{i=1}^{k}b_i(t)\Delta u\big(x,\tau_i(t)\big)$$
$$+\sum_{i=1}^{m}q_i(x,t)u\big(x,\sigma_i(t)\big)=0,\quad (x,t)\in\Omega. \tag{E$_\mathrm{L}$}$$

Theorem 3 *(Linear case). Assume that* (H1)–(H6) *and* (H8) *hold. If the differential inequality*

$$y'(t)+q_j(t)\left(1-\sum_{i=1}^{l}\alpha_i h_i\big(\sigma_j(t)\big)\right)y\big(\sigma_j(t)\big)\leqslant 0 \tag{27}$$

has no eventually positive solution, then every solution u of the problem ($\mathrm{E_L}$), ($\mathrm{B_1}$) *is oscillatory in Ω.*

Proof. The proof follows by using the same arguments as in Theorem 1 and hence will be omitted. □

Theorem 4 *(Linear case). Assume that* (H1)–(H6) *and* (H8) *hold. If the differential inequality* (27) *has no eventually positive solution, then every solution u of* ($\mathrm{E_L}$), ($\mathrm{B_2}$) *is oscillatory in Ω.*

Proof. By the same arguments as were used in Theorem 2, the proof will be omitted. □

In the proof of the subsequent corollaries we shall use the results of Yoshida [6], Shoukaku and Yoshida [7].

Corollary 3 *(Linear case). Assume that* (H1)–(H6), (H8) *and the following*

(H10) *$\sigma_j(t)\leqslant t$ and $\sigma_j(t)$ is nondecreasing for $t\geqslant t_0$.*

If

$$\liminf_{t\to\infty}\int_{\sigma_j(t)}^{t}q_j(s)\left(1-\sum_{i=1}^{l}\alpha_i h_i\big(\sigma_j(t)\big)\right)ds>\frac{1}{e}, \tag{28}$$

then every solution u of the problem ($\mathrm{E_L}$), ($\mathrm{B_1}$) *is oscillatory in Ω.*

Corollary 4 *(Linear case). Assume that* (H1)–(H6), (H8) *and* (H10) *hold. If condition* (28) *holds, then every solution u of the problem* ($\mathrm{E_L}$), ($\mathrm{B_2}$) *is oscillatory in Ω.*

Example 1. Let us consider the problem

$$\frac{\partial}{\partial t}\left(u(x,t)+\frac{1}{2}\tan^{-1}\big(u(x,t+2\pi)\big)\right)-u_{xx}(x,t)-u_{xx}(x,t+\pi)$$
$$+\frac{1}{2}\left(\frac{3+2\sin^2 x\sin^2 t}{1+\sin^2 x\sin^2 t}\right)u\left(x,t-\frac{\pi}{2}\right)=0,\quad (x,t)\in(0,\pi)\times(0,\infty), \tag{29}$$
$$u(0,t)=u(\pi,t)=0,\quad t>0. \tag{30}$$

Here $n = l = k = \tilde{m} = m = 1$, $G = (0, \pi)$, $h_1(t) = \frac{1}{2}$, $\omega_1(z) = \tan^{-1}(z)$, $\alpha_1 = 1$, $\rho_1(t) = t + 2\pi$, $a(t) = b_1(t) = 1$, $\tau_1(t) = t + \pi$,

$$q_1(t) = \frac{1}{2}\left(\frac{3}{1+\sin^2 t}\right), \qquad \varphi_1(z) = \varphi_{11}(z) = \varphi_{12}(z) = z \quad \text{and} \quad \sigma_1(t) = t - \frac{\pi}{2}.$$

Since

$$\int_{t_0}^{\infty} \frac{1}{2}\left(\frac{3}{1+\sin^2 t}\right)\left(1 - \frac{1}{2}\right) dt \geqslant \frac{3}{8}\int_{t_0}^{\infty} dt = \infty,$$

$$\int_{t-\frac{\pi}{2}}^{t} \frac{1}{2}\left(\frac{3}{1+\sin^2 t}\right)\left(1 - \frac{1}{2}\right) dt \geqslant \frac{3}{16}\pi > \frac{1}{e},$$

we conclude from Corollaries 1 and 3 that every solution u of (29), (30) is oscillatory in $(0, \pi) \times (0, \infty)$ or satisfies (2). In fact, $u(x, t) = \sin t \sin x$ is a oscillatory solution.

Example 2. We consider the problem

$$\frac{\partial}{\partial t}\left(u(x,t) + \frac{1}{2}\left(\frac{e^{u(x,t)} - 1}{e^{u(x,t)} + 1}\right)\right) - a(t)u_{xx}(x,t) - b_1(t)u_{xx}\big(x, \tau_1(t)\big)$$
$$+ \frac{e^{-2t} + 5e^{-t} + 5}{(e^{-t} + 2)^3}\big(e^{u(x,t)} - e^{-u(x,t)}\big) = 0, \quad (x,t) \in (0, L) \times (0, \infty), \tag{31}$$
$$-u_x(0,t) = u_x(L,t) = 0, \quad t > 0. \tag{32}$$

Here $n = l = k = 1$, $G = (0, L) \subset \mathbb{R}$, $h_1(t) = \frac{1}{2}$,

$$\omega_1(z) = \frac{e^z - 1}{e^z + 1}, \quad \alpha_1 = \frac{1}{2}, \quad \rho_1(t) = t, \quad q_1(t) = \frac{e^{-2t} + 5e^{-t} + 5}{(e^{-t} + 2)^3}, \quad \sigma_1(t) = t$$

and $\varphi_1(z) = \varphi_{11}(z) = \varphi_{12}(z) = e^z - e^{-z}$. Since

$$\int_{t_0}^{\infty} \frac{e^{-2t} + 5e^{-t} + 5}{(e^{-t} + 2)^3}\left(1 - \frac{1}{4}\right) dt \geqslant \int_{t_0}^{\infty} \frac{5}{36}\, dt = \infty,$$

Corollary 2 implies that every nonoscillatory solution u of (31), (32) satisfies

$$\lim_{t\to\infty} \int_0^L u(x,t)\, dx = 0.$$

In fact, one such solution is $u(x, t) = \ln(e^{-t} + 1)$.

Example 3. Consider the boundary value problem

$$\frac{\partial}{\partial t}\left(u(x,t) + \frac{e}{3}\tan^{-1}\big(u(x,t+1)\big)\right) - u_{xx}(x,t) + \frac{1}{3e(e^{-2t-2}\cos^2 x + 1)}u(x,t-1) = 0,$$
$$(x,t) \in (0, \pi) \times (0, \infty), \tag{33}$$
$$-u_x(0,t) = u_x(\pi,t) = 0, \quad t > 0. \tag{34}$$

Here $n = l = \tilde{m} = m = 1$, $G = (0, \pi)$, $h_1(t) = e/3$, $\omega_1(z) = \tan^{-1}(z)$, $\alpha = 1$, $\rho_1(t) = t + 1$, $q_1(t) = 1/(3e(1 + e^{-2t-2}))$, $\varphi_1(z) = \varphi_{11}(z) = \varphi_{12}(z) = z$ and $\sigma_1(t) = t - 1$. Since

$$\int_{t_0}^{\infty} \frac{1}{3e(1+e^{-2t-2})}\left(1 - \frac{e}{3}\right) dt \geqslant \int_{t_0}^{\infty} \frac{1}{3e(1+e^{-2})}\left(1 - \frac{e}{3}\right) dt = \infty,$$

$$\int_{t-1}^{t} \frac{1}{3e(1+e^{-2t-2})}\left(1 - \frac{e}{3}\right) ds \leqslant \frac{1}{3e} < \frac{1}{e},$$

then Corollary 4 does not apply but Corollary 2 does. Hence, we see that every oscillatory solution u of (33), (34) satisfies

$$\lim_{t\to\infty} \int_0^{\pi} u(x,t)\, dx = 0.$$

In fact, one such solution is $u(x,t) = e^{-t}\cos x$.

4. Linearized oscillation for equation (E)

In this section we consider the equation

$$\frac{\partial}{\partial t}\left(u(x,t) + \sum_{i=1}^{l} h_i(t)\omega_i\big(u\big(x, \rho_i(t)\big)\big)\right) - a(t)\Delta u(x,t) - \sum_{i=1}^{k} b_i(t)\Delta u\big(x, \tau_i(t)\big)$$
$$+ \sum_{i=1}^{m} q_i(t)\varphi_i\big(u\big(x, \sigma_i(t)\big)\big) = 0, \quad (x,t) \in \Omega, \tag{E$_1$}$$

where $\omega_i(s) \in C(\mathbb{R}; \mathbb{R})$ $(i = 1, 2, \ldots, l)$, $q_i(t) \in C([0,\infty); [0,\infty))$, $\sigma_i(t) \in C([0,\infty); [0,\infty))$, $\lim_{t\to\infty} \sigma_i(t) = \infty$ and $\varphi_i(\xi) \in C(\mathbb{R}; \mathbb{R})$ $(i = 1, 2, \ldots, m)$.

We assume that:

(H11) there exist positive constants $1 < \tilde{\alpha}$ and $0 < \tilde{\beta} < 1$ such that

$$\begin{cases} \tilde{\alpha} u \geqslant \omega_i(u) \geqslant 0 & \text{if } 0 < u < \delta, \\ 0 \geqslant \omega_i(u) \geqslant \tilde{\alpha} u & \text{if } -\delta > u > 0 \end{cases} \quad \text{and} \quad \begin{cases} \varphi_i(u) \geqslant \tilde{\beta} u & \text{if } 0 < u < \delta, \\ \tilde{\beta} u \geqslant \varphi_i(u) & \text{if } -\delta > u > 0, \end{cases}$$

where δ is positive number;

(H12) $\tilde{\alpha} \sum_{i=1}^{l} h_i(t) \leqslant 1$.

Theorem 5. *Assume that* (H1), (H2), (H6), (H11) *and* (H12) *hold. If the differential inequality*

$$y'(t) + \tilde{\beta} q_j(t)\left(1 - \tilde{\alpha} \sum_{i=1}^{l} h_i\big(\sigma_j(t)\big)\right) y\big(\sigma_j(t)\big) \leqslant 0 \tag{35}$$

has no eventually positive bounded solution, then every bounded solution u of the problem (E$_1$), (B$_1$) *is oscillatory in Ω.*

Proof. Suppose that there is a bounded nonoscillatory solution u of (E_1), (B_1). We may assume that $\delta > u > 0$ in $G \times [t_0, \infty)$ for some $t_0 > 0$. Since (H2) holds, we see that $\delta > u(x, \rho_i(t)) > 0$ $(i = 1, 2, \dots, l)$, $\delta > u(x, \tau_i(t)) > 0$ $(i = 1, 2, \dots, k)$ and $\delta > u(x, \sigma_i(t)) > 0$ $(i = 1, 2, \dots, m)$ in $G \times [t_1, \infty)$ for some $t_1 \geqslant t_0$. From (H11) we obtain

$$\frac{\partial}{\partial t}\left(u(x,t) + \sum_{i=1}^{l} h_i(t)\omega_i\big(u\big(x, \rho_i(t)\big)\big)\right) - a(t)\Delta u(x,t) - \sum_{i=1}^{k} b_i(t)\Delta u\big(x, \tau_i(t)\big)$$
$$+ \tilde{\beta}\sum_{i=1}^{m} q_i(t)u\big(x, \sigma_i(t)\big) \leqslant 0, \quad t \geqslant t_1. \tag{36}$$

Proceeding as in the proof of Theorem 1, we obtain

$$z'(t) + \tilde{\beta} q_j(t) V\big(\sigma_j(t)\big) \leqslant 0, \quad t \geqslant t_1, \tag{37}$$

for some $j \in \{1, 2, \dots, m\}$, where

$$z(t) = \int_G z(x,t)\Phi(x)\,dx, \qquad V(t) = \int_G u(x,t)\Phi(x)\,dx.$$

Therefore $z(t)$ is nonincreasing. It follows from (4) and (H11) that

$$z(x,t) \leqslant u(x,t) + \tilde{\alpha}\sum_{i=1}^{l} h_i(t)u\big(x, \rho_i(t)\big).$$

Multiplying the above inequality by $\Phi(x)$ and then integrating over G yields

$$V(t) \geqslant z(t) - \tilde{\alpha}\sum_{i=1}^{l} h_i(t)V\big(\rho_i(t)\big) \geqslant \left(1 - \tilde{\alpha}\sum_{i=1}^{l} h_i(t)\right)z(t), \quad t \geqslant t_1. \tag{38}$$

Combining (37) with (38) yields

$$z'(t) + \tilde{\beta}\sum_{i=1}^{m} q_i(t)\left(1 - \tilde{\alpha}\sum_{i=1}^{l} h_i\big(\sigma_j(t)\big)\right)z\big(\sigma_j(t)\big) \leqslant 0, \quad t \geqslant t_1.$$

Hence, $z(t)$ is a positive bounded solution of (35) in $[t_1, \infty)$. This is a contradiction.

If $0 > u > -\delta$ in $G \times [t_0, \infty)$, then we have

$$z_t(x,t) - a(t)\Delta v(x,t) - \sum_{i=1}^{k} b_i(t)\Delta v\big(x, \tau_i(t)\big) + \tilde{\beta}\sum_{i=1}^{m} q_i(t)v\big(x, \sigma_i(t)\big) \leqslant 0, \quad t \geqslant t_1,$$

and

$$z(x,t) = v(x,t) + \sum_{i=1}^{l} h_i(t)\omega_i\big(v\big(x, \rho_i(t)\big)\big),$$

where $v \equiv -u$. Proceeding as in the case where $u > 0$, we are led to a contradiction. The proof is complete. $\square$

Theorem 6. *Assume that* (H1), (H2), (H6), (H11) *and* (H12) *hold. If the differential inequality* (35) *has no eventually positive bounded solution, then every bounded solution u of the problem* (E_1), (B_2) *is oscillatory in Ω.*

Corollary 5. *Assume that* (H1), (H2), (H6), (H11) *and* (H12) *hold. If*

$$\int_{\sigma_j(t)}^{t} \tilde{\beta} q_j(s)\left[1-\tilde{\alpha}\sum_{i=1}^{l} h_i\big(\sigma_j(s)\big)\right] ds > \frac{1}{e}, \tag{39}$$

then every bounded solution u of the problem (E_1), (B_1) *is oscillatory in* Ω.

Corollary 6. *Assume that* (H1), (H2), (H6), (H11) *and* (H12) *hold. If* (39) *holds, then every bounded solution u of the problem* (E_1), (B_2) *is oscillatory in* Ω.

5. Oscillations in neutral logistic equations

In this section we extend the logistic equation $(*)$ to the following:

$$\frac{\partial N(x,t)}{\partial t} = N(x,t)\left\{\sum_{i=1}^{m} q_i(t)\left(1-\frac{N(x,t-\sigma_i)}{K}\right)+\sum_{i=1}^{l} h_i\cdot\frac{\partial}{\partial t}\left(1-\frac{N(x,t+\rho_i)}{K}\right)\right\},$$
$$(x,t)\in\Omega. \tag{E_2}$$

By introducing the change of variable

$$u(x,t)=\ln\left(\frac{N(x,t)}{K}\right), \quad K>0,$$

Eq. (E_2) is transformed to

$$\frac{\partial}{\partial t}\left(u(x,t)+\sum_{i=1}^{l} h_i\cdot\big(e^{u(x,t+\rho_i)}-1\big)\right)+\sum_{i=1}^{m} q_i(t)\big(e^{u(x,t-\sigma_i)}-1\big)=0, \quad (x,t)\in\Omega, \tag{E_3}$$

where h_i (<1), ρ_i $(i=1,2,\ldots,l)$ and σ_i $(i=1,2,\ldots,m)$ are positive constants. We establish the sufficient conditions for oscillation of every positive and bounded solution (E_2) about the unique positive steady K. Clearly, every solution of (E_2) oscillates about K if and only if every solution of (E_3) oscillates.

Theorem 7. *If*

$$\int_{t-\sigma_j}^{t} \tilde{\beta} q_j(s)\left[1-\tilde{\alpha}\sum_{i=1}^{l} h_i\right] ds > \frac{1}{e}, \tag{40}$$

then every bounded positive solution of the problem (E_2), (B_1) *oscillates about the positive steady state* K.

Proof. In this case $h_i(t)=h_i$, $\varphi_1(u)=\omega_1(u)=e^u-1$, $\rho_i(t)=t+\rho_i$ and $\sigma_i(t)=t-\sigma_i$. Then Eq. (E_3) can be written in the form (E_2) and the conditions of Corollary 5 are satisfied. The conclusion follows from Corollary 5. □

Theorem 8. *If* (40) *holds, then every bounded positive solution of* (E_2), (B_2) *oscillates about the positive steady K.*

Example 4. We consider the problem

$$\frac{\partial}{\partial t}\left(u(x,t)+\frac{1}{2}\left(e^{u(x,t+2\pi)}-1\right)\right)+\frac{1}{2}\left(e^{u(x,t-5\pi/2)}-1\right)$$
$$+\left(\frac{2}{2+\cos t}\right)\left(e^{u(x,t-5\pi/2)}-1\right)=0,\quad (x,t)\in(0,L)\times(0,\infty), \tag{41}$$
$$-u_x(0,t)=u_x(L,t)=0,\quad t>0. \tag{42}$$

Here $n=1$, $l=1$, $m=2$, $h_1=\frac{1}{2}$, $\rho_1=\frac{\pi}{2}$, $\sigma_1=\sigma_2=\frac{5}{2}\pi$, $q_1(t)=\frac{1}{2}$ and $q_2(t)=\frac{2}{2+\cos t}$. If we choose $\tilde{\alpha}=\frac{3}{2}$, $\tilde{\beta}=\frac{1}{2}$ and $j=1$, then

$$\int_{t-\frac{5}{2}\pi}^{t}\frac{1}{4}\left[1-\frac{3}{4}\right]ds=\frac{5}{32}\pi>\frac{1}{e}.$$

It follows from Corollary 2 that every bounded solution u of (41), (42) is oscillatory in $(0,L)\times(0,\infty)$. For example, $u(x,t)=\ln(1+\frac{1}{2}\cos t)$ is such a solution.

Remark 2. From Example 1 we see that $N(x,t)=1+\frac{1}{2}\cos t$ is oscillatory about $K=1$ and satisfies the following problem:

$$\frac{\partial N(x,t)}{\partial t}=N(x,t)\left\{\frac{1}{2}\left(1-N\left(x,t-\frac{5}{2}\pi\right)\right)+\left(\frac{2}{2+\cos t}\right)\left(1-N\left(x,t-\frac{5}{2}\pi\right)\right)\right.$$
$$\left.+\frac{1}{2}\cdot\frac{\partial}{\partial t}\left(1-N(x,t+2\pi)\right)\right\},$$
$$-N_x(0,t)=N_x(L,t)=0,\quad t>0,$$

Acknowledgment

The author thanks Professor N. Yoshida for valuable advice.

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